PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On the Rotation of the Sun

D. O. Gough

Phil. Trans. R. Soc. Lond. A 1984 **313**, 27-38
doi: 10.1098/rsta.1984.0080

Phil. Trans. R. Soc. Lond. A **313**, 27–38 (1984) [27]
Printed in Great Britain

On the rotation of the Sun

By D. O. Gough

Institute of Astronomy and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Madingley Road, Cambridge CB3 0HA, U.K.

[Plate 1]

An asymptotic method is developed to estimate the rotational splitting of sectoral five-minute solar oscillations. Integral formulae are obtained which can be inverted to yield the variation with depth of the Sun's angular velocity near the equatorial plane. The result is a functional of smoothed data, and does not rely on a detailed theoretical model of the Sun. The method has been tested with artificial data (computed from a theoretical solar model) of a kind similar to some real solar data obtained recently by Duvall & Harvey (*Nature, Lond.* **310**, 19 (1984)). The results are encouraging, for they reproduce at least the broadest feature of the somewhat arbitrary angular velocity with which the theoretical model was endowed. When applied to the real data, the method yields a result similar to that derived by Duvall *et al.* (*Nature, Lond.* **310**, 22 (1984)) by another procedure.

Introduction

It is generally believed that when the Sun arrived on the Main Sequence it was spinning much more rapidly than it does today. Braking by the solar wind caused the outer layers to slow down, but by how much the torque has been transmitted to the deep interior is a matter of some controversy. The amount of angular momentum transfer by Eddington–Sweet or Ekman circulation, by waves, turbulence or magnetic fields is uncertain, and there is a wide range of theoretical estimates of the angular velocity Ω in the deep interior.

Rotation influences the frequencies of free oscillation of the Sun. These can therefore be used as a diagnostic. As different modes of oscillation sample the rotation differently, with a sufficient variety one might hope to be able to estimate how the angular velocity really varies with both latitude and depth.

There are several reports of observations of rotational splitting already in the literature (Claverie *et al.* 1981; Hill *et al.* 1982; Delache & Scherrer 1983). Aside from doubts about their interpretation, the data are not sufficiently extensive to enable one to make a reliable estimate of Ω. However, recently Duvall & Harvey (1984*a*) have observed splitting in sectoral five-minute oscillations of degree ranging from unity to 100. I describe here how one can use these data to estimate Ω near the equatorial plane, giving some indication of the uncertainty in the result. The method requires a knowledge of not only the frequency splittings, but also the absolute frequencies of the oscillations. From the latter sufficient information about the stratification of the Sun can be obtained to permit an inference of Ω that does not rely on the theory of the evolution of the Sun.

The inversion procedure has been done with artificial data constructed from an arbitrarily specified rotation law. The application of the method to the real data is still in a preliminary phase, and is therefore discussed only briefly.

The data of Duvall & Harvey

Absolute frequencies of high-degree five-minute p modes are presented graphically by Duvall (1982). Frequencies of p modes of low and intermediate degree are reported by Duvall & Harvey (1983, 1984*b*). These are zonal modes, such as that illustrated in figure 1*a*, plate 1. Duvall & Harvey (1984*a*) have also measured the frequencies and the splittings of some sectoral modes, similar to those in figures 1*b*–*d*, plate 1. These have been averaged into 37 groups of modes of like degree. The values of the orders and degrees of the modes for which rotational splitting has been observed are listed in table 1.

Ray theory for high-frequency acoustic modes in a non-rotating star

All the modes observed by Duvall & Harvey are essentially high-frequency acoustic modes. Their wavelengths are short compared with the radius of the Sun, and throughout most of the region within which the modes are confined the local wavelengths are also shorter than the scale heights of density and temperature. Therefore the modes should be amenable to asymptotic analysis. Either ray theory or a J.W.K.B. analysis of the normal mode equations (which is similar) can be used. I present here the ray theory.

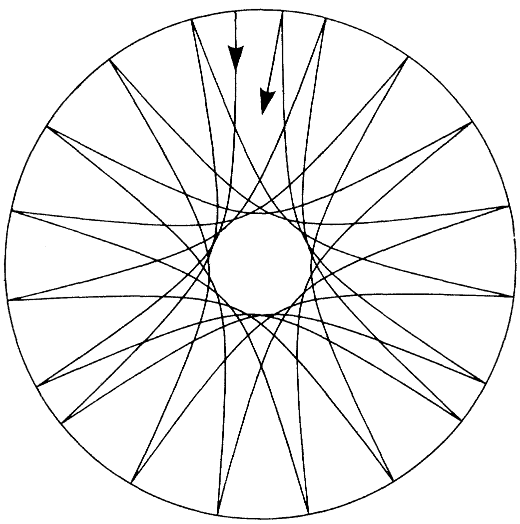


Figure 2. A multiply reflected ray path of a mode with $n/l = 5$. The circle represents the surface of the Sun.

A typical ray path is illustrated in figure 2. It may be considered to lie in the equatorial plane, and therefore most simply represents the sectoral modes of high degree, which are concentrated near the equator (figure 1*d*). A sound wave propagating obliquely downwards into the Sun experiences a rise in sound speed. It is therefore refracted, and returns to the more highly stratified surface layers. There it is reflected down; as Lamb (1908) has pointed out, vertical propagation in a stratified atmosphere is impossible where the density scale height is less than some critical value comparable with the wavelength of the wave. Evidently in the reflecting layer both ray theory and J.W.K.B. analysis break down.

Notwithstanding this inconsistency near the surface I assume the validity of a local dispersion

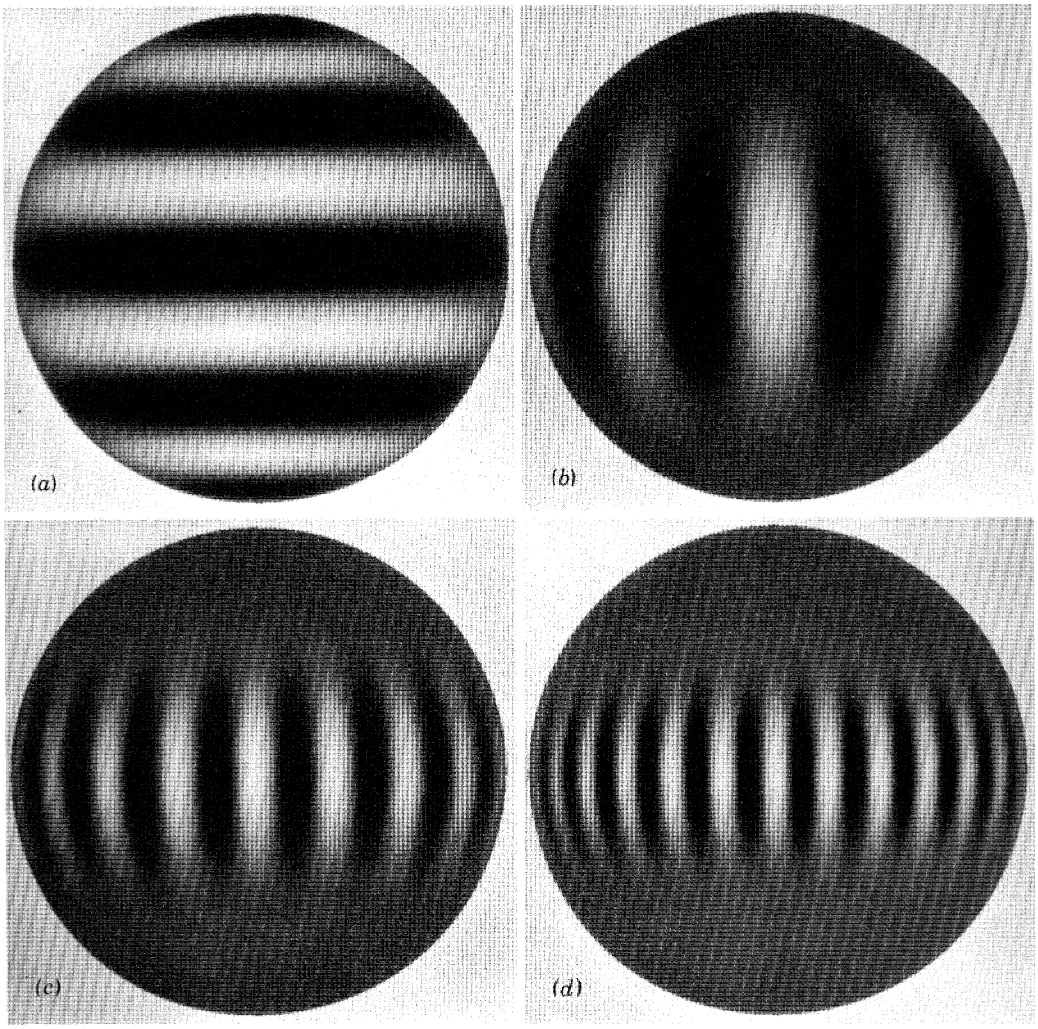


FIGURE 1. Contributions to the line-of-sight velocity at the surface of the Sun from a selection of five-minute oscillation modes. The grey level represents the magnitude of the velocity: black is towards and white is away from the observer (or vice-versa). The motion is almost vertical, so the mid-grey at the edges of the Sun's image represents zero velocity. (*a*) is the zonal mode ($m = 0$) of degree $l = 10$, and (*b*)–(*d*) are sectoral modes ($m = l$) of degrees 10, 20 and 30 respectively.

(*Facing p.* 28)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

relation, which relates frequency ω to the wave number $\boldsymbol{k}$ at all points $\boldsymbol{r}$ in the region of propagation:

$$\omega = W(\boldsymbol{k}, \boldsymbol{r}), \tag{1}$$

and refer all quantities to spherical polar coordinates (r, θ, ϕ) about the axis through the centre of the Sun perpendicular to the equatorial plane. In particular, $\boldsymbol{k} = (k_r, 0, k_\phi)$. Since the equilibrium structure of the Sun is independent of ϕ it follows from (1) that rk_ϕ is constant along a ray path (cf. Whitham 1974). At any instant, therefore, the phase ψ varies linearly with ϕ. The wave form is proportional to $\sin m\phi$, where m is a constant.

Resonant oscillations, or modes, are produced when multiply refracted and reflected waves interfere constructively, just as in the simpler case of an organ pipe. This occurs only for a discrete set of frequencies. To compute those frequencies consider the segment ABC of the ray illustrated in figure 3, with $\Omega = 0$. Constructive interference occurs when the phase that is

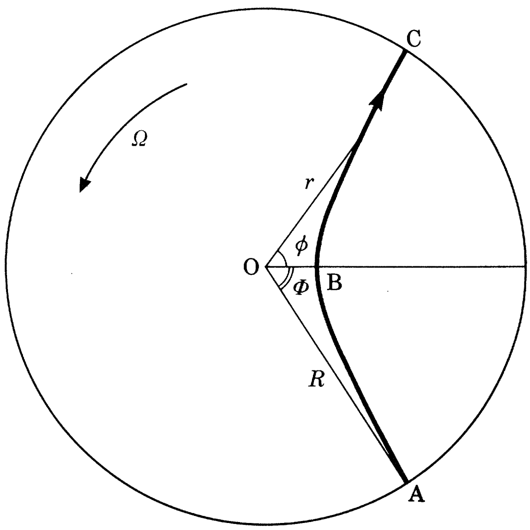


FIGURE 3. A single ray segment, and the coordinates used to define a point on it.

transported from A to C along the ray coincides with that of the interference pattern on the surface. Since the interference pattern is formed by a superposition of waves all of which vary with ϕ like $\sin m\phi$, the pattern must vary with ϕ likewise. It follows that m must be an integer (to obtain a period of 2π), which is identified with the azimuthal order (and the degree) of the sectoral mode. Moreover the phase difference between A and C is simply $2m\Phi$, where 2Φ is the angle subtended by A and C about the centre O of the star.

If the acoustically reflecting surface layers of the Sun were rigid, the phase change $\Delta\psi$ along the resonant ray would differ from $m\Phi$ by an integral multiple n of 2π. However, in reality, there is a phase jump $(2\pi\alpha)$ on reflection, which leads to

$$\Delta\psi = \int_{\mathrm{A}}^{\mathrm{C}} \boldsymbol{k}\cdot \mathrm{d}\boldsymbol{r} = 2m\Phi + 2\pi(n+\alpha), \tag{2}$$

where the integral is taken along the ray path. The quantity α depends on the mass of the relatively inert atmosphere above the reflecting layer. Because the waves propagate nearly vertically near where they are reflected, its value is independent of m. It is determined predominantly by the mean polytropic index in the reflecting layers.

Deep in the interior those effects of stratification that cause reflection in the surface layers are negligible (cf. Deubner & Gough 1984), and the local dispersion relation for the waves reduces to the usual weak-stratification limit:

$$\omega^4-(k^2c^2+N^2)\,\omega^2+k_\phi^2\,c^2N^2=0 \tag{3}$$

(see, for example, Eckart 1960), where $k^2=k_r^2+k_\phi^2\approx k_r^2+(m/r)^2$ is the square of the total wavenumber, c is the sound speed and N is the buoyancy frequency. Differentiating with respect to the components of $\boldsymbol{k}$ yields

$$(\omega^2-N^2c^2k_\phi^2/\omega^2)\,\omega(\partial\omega/\partial k_r,0,\partial\omega/\partial k_\phi)=c^2[(\omega^2-N^2)\,k_\phi,0,\omega^2k_r], \tag{4}$$

from which follows the equation of the ray path:

$$r\,\mathrm{d}\phi/\mathrm{d}r=(1-N^2/\omega^2)\,k_\phi/k_r=[(1-N^2/\omega^2)/(w^2/a^2-1)]^{\frac{1}{2}}, \tag{5}$$

where $w\equiv\omega/m$ and $a\equiv c/r$. The phase change along the ray segment can therefore be written

$$\Delta\psi=2\int_{\mathrm{B}}^{\mathrm{C}}\left(k_r+rk_\phi\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\mathrm{d}r, \tag{6}$$

and the polar angle subtended is

$$2\Phi=2\int_{\mathrm{B}}^{\mathrm{C}}\frac{\mathrm{d}\phi}{\mathrm{d}r}\,\mathrm{d}r. \tag{7}$$

Substituting (6) and (7) into the resonance condition (2) and using (5) to define the ray path, yields the equation that determines the oscillation eigenfrequencies:

$$\frac{\pi(n+\alpha)}{\omega}=\int_{a=\omega}^{a=a_\mathrm{s}}a^{-1}[(1-N^2/\omega^2)\,(1-a^2/w^2)]^{\frac{1}{2}}\,\mathrm{d}\ln r. \tag{8}$$

Strictly speaking the upper limit of integration should be at the level of reflection. However, since five-minute modes are reflected quite near the surface of the Sun, I shall replace a_s by the value of a in the photosphere. It is a property of theoretical models of the Sun that the quantity $a(r)$ is a monotonic decreasing function of r. It is likely that that is also true of the real Sun. Therefore there is a unique ray with given values of ω and w joining A to C, and none of the multiplicity difficulties that are sometimes encountered in geoseismology need be considered.

For five-minute acoustic modes, $\omega/2\pi\approx 3$ mHz and $N/2\pi\lesssim 0.4$ mHz, so buoyancy makes a correction of only 1% to the integral in (8). The integral may therefore be expanded about $N=0$, yielding

$$\frac{\pi(n+\alpha)}{\omega}+\frac{1}{\omega^2}\Psi\approx F(w)\equiv\int_{\ln r_\mathrm{t}}^{\ln R}a^{-1}(1-a^2/w^2)^{\frac{1}{2}}\,\mathrm{d}\ln r, \tag{9}$$

where

$$\Psi(w)\equiv\frac{1}{2}\int_{\ln r_\mathrm{t}}^{\ln R}a^{-1}N^2(1-a^2/w^2)^{\frac{1}{2}}\,\mathrm{d}\ln r, \tag{10}$$

$a(r_\mathrm{t})=w$ and R is the radius of the Sun.

The high-degree modes, which have small values of w, are confined to the convection zone, where $N \approx 0$. Consequently $\Phi \approx 0$, and $\pi(n+\alpha)/\omega$ is a function of w alone, a result that was first noticed by Duvall (1982) in the real solar data. Notice now that since α is independent of m it can be determined by comparing the functional form of (9) and (10) with the solution for high-degree modes in a polytropic envelope of index μ (see, for example, Gough 1978). That yields $\alpha = \frac{1}{2}\mu$. Since the waves cannot experience the detailed structure of the Sun above the reflecting layer, μ must be interpreted as the characteristic polytropic index at the level of reflection. (See, for example, Vandakurov (1967) and Tassoul (1980), who consider asymptotically high-order acoustic modes of low degree by the J.W.K.B. method.)

Notice that the lower limit of integration is the radius r_t at which $a = w$. Here $\mathrm{d}r/\mathrm{d}\phi = 0$ and the ray is horizontal; the waves, which travel at the local sound speed c save for the small modification by buoyancy, have an angular velocity about the centre of the Sun that is approximately equal to the angular phase speed w of the interference pattern that is measured at the surface.

Ray theory for a rotating star

The cyclic angular velocity $\Omega_s/2\pi$ of the surface of the Sun is about 0.45 μHz at the equator, and therefore influences the frequencies of acoustic waves to a much lesser extent than buoyancy. It is therefore sufficient at present to consider only those corrections to the eigenfrequency equations (8), or (9) and (10), arising from the interior angular velocity $\Omega(\boldsymbol{r})$, that are linear in Ω/ω.

It is straightforward to show that the effect of a small rotation on the local dispersion relation (3) is quadratic in Ω (see, for example, Eckart 1960; Whitham 1974). Thus it appears that the only local linear perturbation to the oscillations is caused by advection by the rotational velocity of the equilibrium state. Consequently with respect to an inertial frame of reference the horizontal component of the group velocity, which is given by (4), is simply augmented by $r\Omega$. Equation (5) for the ray path then becomes

$$r\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{(1-N^2/\omega^2)\,k_\phi + (\omega^2 - N^2c^2k_\phi^2/\omega^2]\,\omega^{-1}c^{-2}r\Omega}{(1-N^2/\omega^2)^{\frac{1}{2}}\,(\omega^2/c^2-k_\phi^2)^{\frac{1}{2}}}. \tag{11}$$

There is a horizontal stretching of the waves which cancels the change in the length of the ray path, leaving the phase change $\Delta\psi$ unmodified. However Φ is altered, as is evident from (7) and (11), and changes the eigenvalue equation to

$$\frac{\pi(n+\alpha)}{\omega} = \int_{\ln r_t}^{\ln R} a^{-1}[(1-N^2/\omega^2)\,(1-a^2/w^2)]^{\frac{1}{2}}\left\{1+\frac{[1-N^2a^2/(\omega^2w^2)]\,\Omega}{w(1-N^2/\omega^2)\,(1-a^2/w^2)}\right\}\mathrm{d}\ln r. \tag{12}$$

Observe that since this equation was derived for sectoral modes, Ω is sampled essentially in the equatorial plane.

The influence of rotation is most usefully expressed as a perturbation to the corresponding frequency of a non-rotating star. Setting

$$\omega = \omega_0 + \omega_1, \tag{13}$$

where ω_0 is the value of ω determined by (8), and expanding (14) to first order in Ω/ω_0 yields

$$\omega_1 = m\int_{\ln r_t}^{\ln R} K\Omega\,\mathrm{d}\ln r \Bigg/ \int_{\ln r_t}^{\ln R} K\,\mathrm{d}\ln r, \tag{14}$$

where

$$K(r, w) = \frac{1 - N^2a^2/(\omega^2w^2)}{a(1-N^2/\omega^2)^{\frac{1}{2}}(1-a^2/w^2)^{\frac{1}{2}}}. \tag{15}$$

Since the errors in the measured rotational splittings ω_1 are much greater than the influence of buoyancy, it is expedient to ignore N^2 in the splitting formula, which gives

$$\int_{\ln r_t}^{\ln R} a^{-1}(1-a^2/w^2)^{-\frac{1}{2}}\Omega \,\mathrm{d}\ln r \equiv G(w) \approx \frac{\omega_1}{m}\int_{\ln r_t}^{\ln R} a^{-1}(1-a^2/w^2)^{-\frac{1}{2}}\mathrm{d}\ln r. \tag{16}$$

This equation predicts that for high-frequency acoustic modes ω_1/m is a function of w alone.

Finally, it should be appreciated that ray theory is valid only when wavenumbers are large. This implies, in particular, that m must not be too small. For sectoral modes $m = l$, where l is the degree of the mode. And from the properties of spherical harmonics we know that the total horizontal wavenumber, upon which ω_0 depends, is really L/r, where $L^2 = l(l+1)$. Therefore wherever w occurs explicitly in the formulae I shall assume it to be defined as ω/L rather than ω/m.

Inversion of the asymptotic formulae

It has already been pointed out that $\Psi \approx 0$ when w is less than the value of a at the base of the convection zone. Then $\pi(n+\alpha)/\omega$ determines $F(w)$. The frequency ω_0 can be measured, and n can be inferred by counting ridges in the k–ω diagram (see, for example, Deubner 1975). As Duvall (1982) has shown, the constant α can then be determined by plotting $\pi(n+\alpha)/\omega$ against w, which is also a measured quantity. For only one value of α do the ridges collapse on to a single curve.

At higher w the ridges in the k–ω diagram no longer collapse precisely on to a line when plotted in Duvall's way, partly because Ψ, though small, is sufficient to destroy the simple relation. It is necessary to add a correction with the appropriate functional form to $\pi(n+\alpha)/\omega$ to engineer coincidence of the data. The resultant is $F(w)$. The small correction measures Ψ, rather inaccurately, from which in principle $N^2(r)$ could be estimated from (10) once $a(r)$ is known. In practice the estimation might be difficult, because the magnitude of the errors in the asymptotic ray theory may be comparable with the contribution from buoyancy.

The function $a(r)$ is obtained by inverting (9). Differentiation with respect to w followed by the substitution $a^{-2} = \xi, w^{-2} = u$ casts (9) into the form of Abel's integral equation, namely

$$\int_u^{a_s^{-2}} \frac{f\,\mathrm{d}\xi}{(\xi-u)^{\frac{1}{2}}} = -2\frac{\mathrm{d}F}{\mathrm{d}w}, \tag{17}$$

where

$$f = -(a^3/2r)\,\mathrm{d}a/\mathrm{d}r \tag{18}$$

can be regarded as an implicit function of ξ. Equations (17) and (18) can therefore be inverted formally to yield, after transforming to the original variables, the value of r at which $a = c/r$ is equal to the observed phase speed w:

$$r = R\exp\left\{-\frac{2}{\pi}\int_{a_s}^{a}(w^{-2}-a^{-2})^{-\frac{1}{2}}\frac{\mathrm{d}F}{\mathrm{d}w}\mathrm{d}w\right\}. \tag{19}$$

This implicitly determines $a(r)$.

Once a is known, $G(w)$ is determined from the splitting data ω_1. Transforming to the variables (ξ, u) converts (16) directly into Abel's form, which can be inverted to yield

$$\Omega(r) = -\frac{2a^2}{\pi}\frac{\mathrm{d}}{\mathrm{d}\ln r}\int_{a_s}^{a} w^{-3}(w^{-2}-a^{-2})^{-\frac{1}{2}}G(w)\,\mathrm{d}w. \tag{20}$$

Equation (10) is inverted in the same way.

Formally, (19) and (20) determine $\Omega(r)$ in terms of only measured quantities, namely $F(w)$, which is inferred from the absolute frequencies $\omega(n, m)$ of the oscillations, and $G(w)$, which is determined in terms of the splitting frequencies ω_1 by (16) once(19) has been solved. The inversion of the data does not refer to any specific model of the Sun, and therefore does not depend on the theory of stellar evolution.

In practice, of course, no finite amount of data can ever specify uniquely the functions F and G. To perform the inversions therefore requires an assumption: one of smoothness is adopted here, and was implemented by fitting splines to the data in the manner discussed by Reinsch (1967). To represent G, say, a smoothing function $g(w)$ is chosen so as to minimize

$$S \equiv \int_{w_A}^{w_B}\left(\frac{\mathrm{d}^2 g}{\mathrm{d}w^2}\right)^2 \mathrm{d}w, \tag{21}$$

among all functions satisfying

$$\frac{1}{I}\sum_{i=1}^{I}\left(\frac{g(w_i)-G_i}{\epsilon_i}\right)^2 \leqslant E^2, \tag{22}$$

where (w_i, G_i) are the I data pairs (here $I = 37$), and w_A and w_B are the least and greatest values of w_i. Here ϵ_i is an estimate of the error in G_i, which can be obtained from the errors in ω_1 (listed in table 1) once $a(r)$ is known and the integral on the right side of (16) is evaluated. When $E = 1$ the minimization of S under the constraint (22) selects a spline which is constrained to lie within the mean confidence level set by the estimated errors. To evaluate the integrals in (19) and (20) it was also necessary to extrapolate the functions to a_s from the smallest values of w for which there are data ($2.6 \times 10^{-5}\ \mathrm{s}^{-1}$ for F and $1.5 \times 10^{-4}\ \mathrm{s}^{-1}$ for G, which correspond to depths beneath the photosphere of about $5 \times 10^{-3}\,R$ and $8 \times 10^{-2}\,R$ respectively).

It must be realized that with only sectoral modes it is not possible to measure the variation of Ω with θ. The function $\Omega(r)$ inferred from (20) is actually an average of the true angular velocity over a range of low latitudes. The half-width of that range is arccos $[m/(m+1)]^{\frac{1}{2}}$, which decreases with increasing m (see figure 1). Since the range of frequencies is fairly small, w and hence the penetration depth is determined mainly by m. Consequently, a broader average is taken by modes that penetrate more deeply. In practice, therefore, if the so-called equatorial acceleration persists well beneath the photosphere, as the motions of sunspots, for example, suggest is the case, there will be a systematic tendency for the function Ω obtained by (20) to underestimate the equatorial angular velocity by more and more at greater and greater depths; this is a result partly of the slower rotation at high latitudes in even the outer layers of the Sun. Of course one could try to correct this. But it must be realized that to do so would require yet another untested assumption, this time about the dependence of Ω on θ at depth. The latitudinal dependence of Ω could be inferred if splitting frequencies corresponding to all the tesseral harmonics were available.

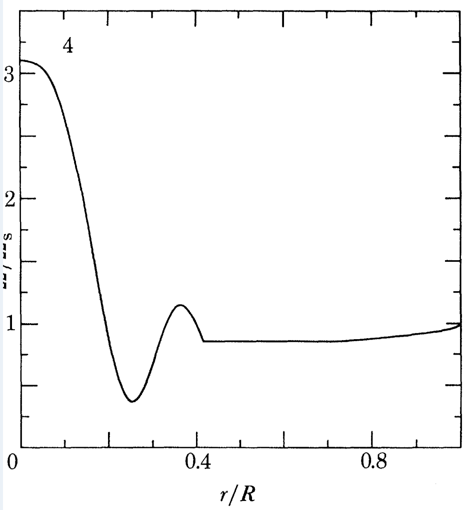
A test of the inversion procedure

A test of the method has been performed with artificial data, constructed from a known model of the Sun. The object was to discover how well the rotation that had been attributed to the model could be recovered from the splitting data. Only mock Duvall–Harvey data have been used. As a result, the results of the test depend in part on the extent of the real data that are available.

The function F was determined from the eigenfrequencies of 2783 theoretical modes corresponding to the five-minute oscillations that have been discussed by Duvall (1982) and Duvall & Harvey (1983, 1984*b*). The identities of the modes were kindly supplied by Duvall & Harvey. Their eigenfrequencies were computed and kindly supplied by J. Christensen-Dalsgaard, using his Model 1 (1982) of the Sun.

The function G was computed from the same 37 averages of the 180 modes that Duvall & Harvey list, but with theoretical splittings. Thus the averaged modes have been treated as true modes. They were calculated from Model A of Christensen-Dalsgaard *et al.* (1979). They were obtained not from the asymptotic formula quoted here but from the exact first-order formula given by Gough (1981) for rotational splitting produced by a latitudinally independent angular velocity with eigenfunctions computed numerically from the full linearized adiabatic pulsation equations. It is not important that a different solar model was used at this stage, since rotational splitting of five-minute p modes is not sensitive to fine details of the model.

The angular velocity Ω that was used to compute the splitting frequencies is displayed in figure 4. The variation with r in the convection zone was determined from the observed latitudinal variation at the surface, under the assumption that at low and mid-latitudes rotation



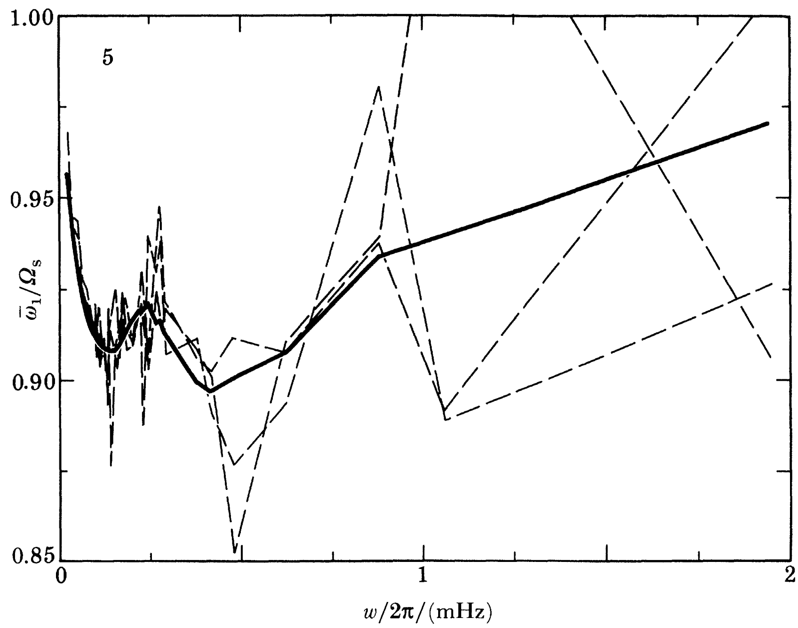


FIGURE 4. The artificial angular velocity that was imposed on the solar model, measured in units of the surface value.

FIGURE 5. The mean splitting frequencies $\bar{\omega}_1$ listed in table 1 resulting from the angular velocity Ω shown in figure 4. They are plotted against $w = [l(l+1)]^{-\frac{1}{2}}\bar{\omega}_0$ and connected by continuous straight lines. The dashed lines connect sets of splitting frequencies obtained by adding Gaussian-distributed errors to $\bar{\omega}_1$.

Table 1. Modes used for angular velocity inversion

Each value of $w/2\pi$ is the uniformly weighted mean of the values of the phase speeds of the modes listed in the previous column plus or minus one standard deviation. The splitting frequency $\bar{\omega}_1$ is the mean value of the splittings computed from the angular velocity in figure 4, and is quoted in units of the Sun's equatorial surface angular velocity Ω_s. The errors are Duvall & Harvey's (1984*a*) error estimates for the real data, expressed as a percentage of Ω_s. Notice that all the values of $\bar{\omega}_1$ are less than unity, even though the core is presumed to be rotating substantially faster than the surface.

m	n	$w/2\pi/(\text{mHz})$	$\bar{\omega}_1$	error (%)
1	17, 20	1.95 ± 0.14	0.971	37
2	13, 18, 19	1.05 ± 0.15	0.940	15
3	18–20, 22	0.880 ± 0.057	0.934	11
4	14–18, 20, 21	0.618 ± 0.072	0.908	4.2
5	12–17, 22	0.472 ± 0.076	0.901	4.8
6	13, 16–18	0.415 ± 0.039	0.897	3.3
7	13, 15, 18, 20	0.374 ± 0.050	0.900	2.2
8	12, 14, 16, 20	0.318 ± 0.048	0.916	7.7
9	11–13, 15, 17, 18, 21	0.286 ± 0.048	0.913	1.5
10	11–14, 16, 19, 22	0.263 ± 0.049	0.916	2.4
11	13, 14, 16, 18	0.243 ± 0.023	0.921	2.6
12	12, 13, 17, 19	0.227 ± 0.032	0.919	3.1
13	12, 14–18, 21	0.222 ± 0.028	0.919	0.4
14	12–15, 17, 19	0.198 ± 0.023	0.915	1.3
15	12–17	0.183 ± 0.016	0.912	1.8
16	11, 13–17	0.173 ± 0.017	0.910	2.4
17	10–13, 15–17	0.157 ± 0.020	0.909	1.5
18	13, 14, 17	0.160 ± 0.013	0.908	0.7
19	11, 14, 16, 17	0.152 ± 0.017	0.908	2.4
20	9, 11–13, 15–17	0.138 ± 0.019	0.908	2.4
21	9, 11–13, 15, 16, 18	0.134 ± 0.020	0.908	1.3
22	9–18	0.130 ± 0.019	0.908	1.3
23	8, 12, 13, 18	0.121 ± 0.023	0.909	0.7
24	9, 10, 13, 14	0.110 ± 0.013	0.910	1.8
25	8–13, 15	0.105 ± 0.013	0.911	1.3
26	9, 10, 12–14, 17	0.109 ± 0.015	0.910	1.1
27	8, 10–12	0.094 ± 0.008	0.913	0.4
28	8–10, 14, 15	0.097 ± 0.015	0.912	0.9
29	11–13	0.099 ± 0.004	0.911	1.5
30	11–13	0.096 ± 0.004	0.911	0.4
40	8, 9, 11–13	0.073 ± 0.008	0.919	1.1
50	9–11	0.060 ± 0.003	0.925	0.4
60	6, 8, 10, 11	0.049 ± 0.006	0.932	1.1
70	5, 7–9	0.040 ± 0.004	0.939	1.1
80	4–7, 9	0.034 ± 0.005	0.944	0.9
90	4,7	0.030 ± 0.004	0.950	0.9
100	4	0.024	0.957	2.2

is a function only of distance from the axis. Such a behaviour is exhibited, for example, in some realizations of the theoretical convection models of Glatzmeier & Gilman (1982). Rotation is rigid throughout part of the radiative region beneath, as would be expected if a magnetic field were present. The core was made to rotate somewhat more rapidly than the rest of the Sun, in accordance with quite common belief; and the oscillatory variation between core and envelope was invented quite arbitrarily to provide some structure to test the resolving power of the inversion. The mean splitting frequencies $\bar{\omega}_1$ are listed in table 1.

To test the prediction that the splitting frequencies depend on w alone, $\bar{\omega}_1$ is plotted against w in figure 5. The points are joined with continuous straight lines. It is apparent that the frequencies do fall roughly on a single curve, which clearly mimics in a distorted way the behaviour of the original rotation curve from which they were derived. The obvious but slight excursions from the curve result probably from the error in regarding an average over several modes as though it were itself a true oscillation mode.

Inversion of the data was performed in the way described above, except that the function $a(r)$ obtained from (19) was smoothed with a spline before entering it into (20). Once $a(r)$ is known it is possible to plot the separation between consecutive penetration radii r_t against the mean r_t. This is shown in figure 6. It sets a bound on the spatial resolution that might be obtainable from the error-free data listed in table 1.

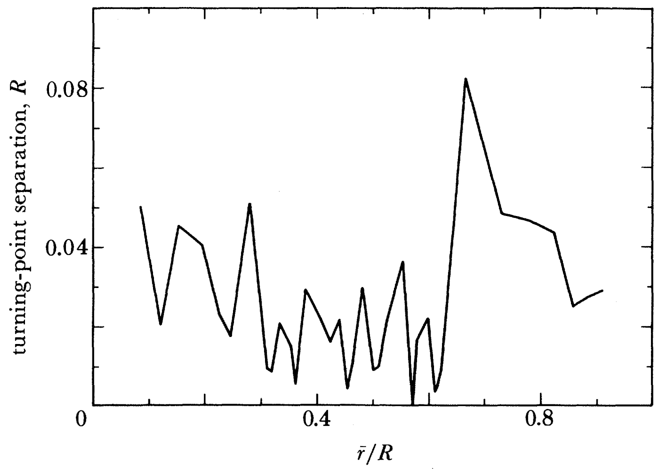
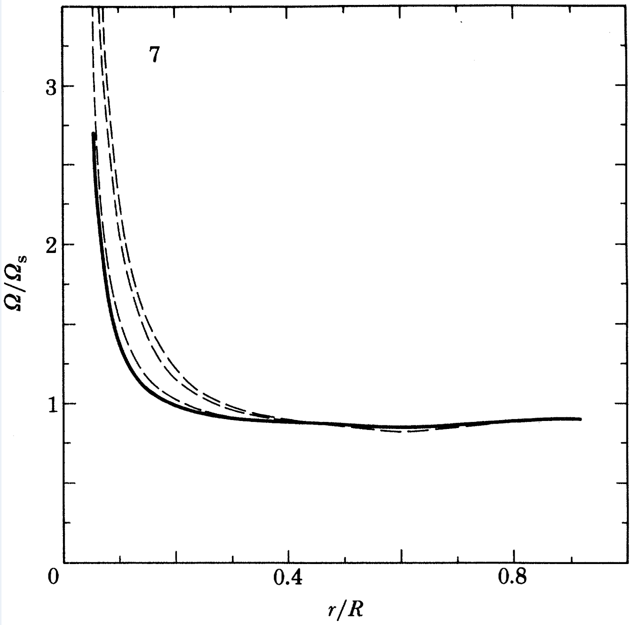


FIGURE 6. Separation $r_t(l_2) - r_t(l_1)$ between adjacent turning points $r_t(l)$ plotted against $\frac{1}{2}[r_t(l_2) + r_t(l_1)]$. Both are measured in units of the solar radius R.

The result of the inversion is shown in figure 7. Broadly speaking it resembles figure 4, though the undulation between $r/R = 0.2$ and $r/R = 0.45$ is absent. Failure to resolve the finer structure in Ω has occurred partly as a result of the spline smoothing that was permitted by criterion (22) with $E = 1$. Setting $E \lesssim 10^{-2}$ yields the result shown in figure 8. Now the undulations are resolved. However their amplitude is underestimated. No doubt the averaging of the splitting frequencies over various values of n at fixed l is responsible for some of the error: the standard deviations of the values of w, listed in table 1, translate into a spatial average over about 0.06 R in that region of the Sun. The central angular velocity cannot be inferred, because none of the mean modes penetrates more deeply than $r = 0.06\ R$.

To obtain some estimate of the reliability of a similar inversion of real data, the procedure was repeated three times with erroneous artificial data. Gaussian distributed noise with zero mean was added to the splitting frequencies $\bar{\omega}_1$, with standard deviations equal to the error estimates quoted by Duvall & Harvey (1984*a*) and listed in table 1. Since the errors are random, quite marked deviations of $\bar{\omega}_1$ from a single curve are now apparent in figure 5. The inferred rotation curves, displayed in figure 7, are superficially similar to the result of inverting the exact data; the splines successfully absorb most of the errors when $E = 1$. If the inversion is repeated with substantially smaller values of E, erroneous oscillations appear in the inferred rotation curve. Their amplitudes exceed the size of the frame of figure 8 when $E = 10^{-2}$; therefore the results of these inversions have not been included in the figure.



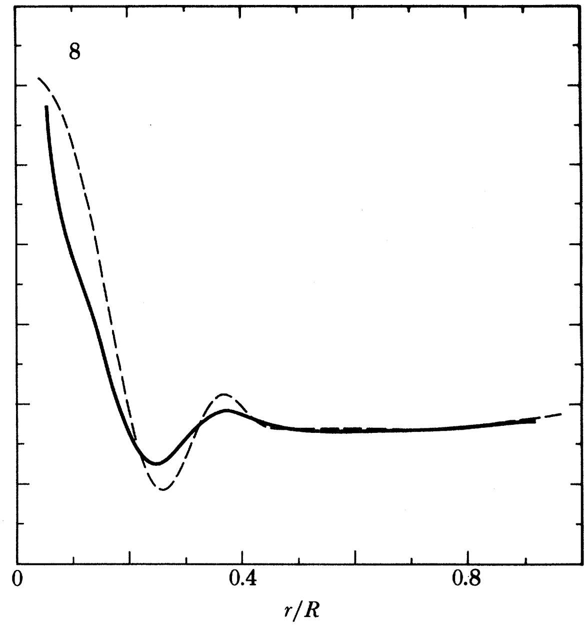


FIGURE 7. The continuous curve is the angular velocity inferred by (19) and (20) from the spline approximation (with $E = 1$) to the splitting frequencies joined by continuous lines in figure 5. The dashed curves were obtained similarly from the erroneous data that are also displayed in figure 5. These results should be compared with the original rotation curve in figure 4.

FIGURE 8. Inversion with $E = 10^{-2}$ of the exact data displayed in figure 5. The dashed curve was obtained by applying a running mean over a distance 0.06 R to the original angular velocity displayed in figure 4.

DISCUSSION

The real solar data of Duvall & Harvey (1984*a*) have also been subjected to the inversion procedure described above. The result is a smooth curve not unlike the histogram in figure 4 of Duvall *et al.* (1984). That was obtained by fitting to the data of the splittings implied by a piecewise constant approximation to Ω. Like the procedure discussed here, this imposes smoothness on the result. As is apparent from the experiment reported here, without a constraint of this kind one cannot progress. It was reported by Duvall *et al.* (1984) that a direct application of the Backus–Gilbert procedure (Backus & Gilbert 1971) to determine resolving power produces poor results, because there are insufficient data to eliminate the large contributions to the splitting from the outer layers of the Sun. Nevertheless it is unlikely that variations of Ω in the convection zone, however small, would happen to be such as to influence in a systematic way a group of oscillation frequencies that happen to have been observed from Earth. Consequently the prejudice of smoothness seems not unreasonable.

I am grateful to J. Christensen-Dalsgaard, T. Duvall and J. Harvey for providing data, and to B. L. N. Kennett for a spline-fitting computer subroutine.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

References

Backus, G. & Gilbert, F. 1970 *Phil. Trans. R. Soc. Lond.* A **266**, 123–192.

Christensen-Dalsgaard, J. 1982 *Mon. Not. R. astr. Soc.* **199**, 735–761.

Christensen-Dalsgaard, J., Gough, D. O. & Morgan, J. G. 1979 *Astron. Astrophys.* **73**, 121–128; **79**, 260.

Claverie, A., Isaak, G. R., McLeod, C. P., van der Raay, H. B. & Roca Cortes, T. 1981 *Nature, Lond.* **293**, 443–445.

Delache, P. & Scherrer, P. H. 1983 *Nature, Lond.* **306**, 651–653.

Deubner, F-L. 1975 *Astron. Astrophys.* **44**, 371–375.

Deubner, F-L. & Gough, D. O. 1984 *A. Rev. Astr. Astrophys.* **22**. (In the press.)

Duvall, T. L. Jr 1982 *Nature, Lond.* **300**, 242–243.

Duvall, T. L. Jr & Harvey, J. W. 1983 *Nature, Lond.* **302**, 24–27.

Duvall, T. L. Jr & Harvey, J. W. 1984*a* *Nature, Lond.* **310**, 19–22.

Duvall, T. L. Jr & Harvey, J. W. 1984*b* *Solar seismology from Space* (ed. R. K. Ulrich) Washington D.C.: N.A.S.A. (In the press.)

Duvall, T. L. Jr, Dziembowski, W. A., Goode, P. R., Gough, D. O., Harvey, J. W. & Leibacher, J. W. 1984 *Nature, Lond.* **310**, 22–25.

Eckart, C. 1960 *Hydrodynamics of oceans and atmospheres*. London: Pergamon.

Glatzmeier, G. & Gilman, P. A. 1982 *Astrophys. J.* **256**, 316–330.

Gough, D. O. 1978 *Proc. EPS Workshop on solar rotation* (ed. G. Belvedere and L. Paternò), pp. 255–268. Catania University Press.

Gough, D. O. 1981 *Mon. Not. R. astr. Soc.* **196**, 731–745.

Hill, H. A., Bos, R. & Goode, P. R. 1982 *Phys. Rev. Lett.* **49**, 1794–1797.

Lamb, H. 1908 *Proc. Lond. math. Soc.* **7**, 122–141.

Reinsch, C. H. 1967 *Numerische Mathematik* **10**, 177–183

Tassoul, M. 1980 *Astrophys. J. Suppl.* **43**, 469–490.

Vandakurov, Yu. V. 1967 *Astr. Zh.* **44**, 786–797.

Whitham, G. B. 1974 *Linear and nonlinear waves*. New York: Wiley.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

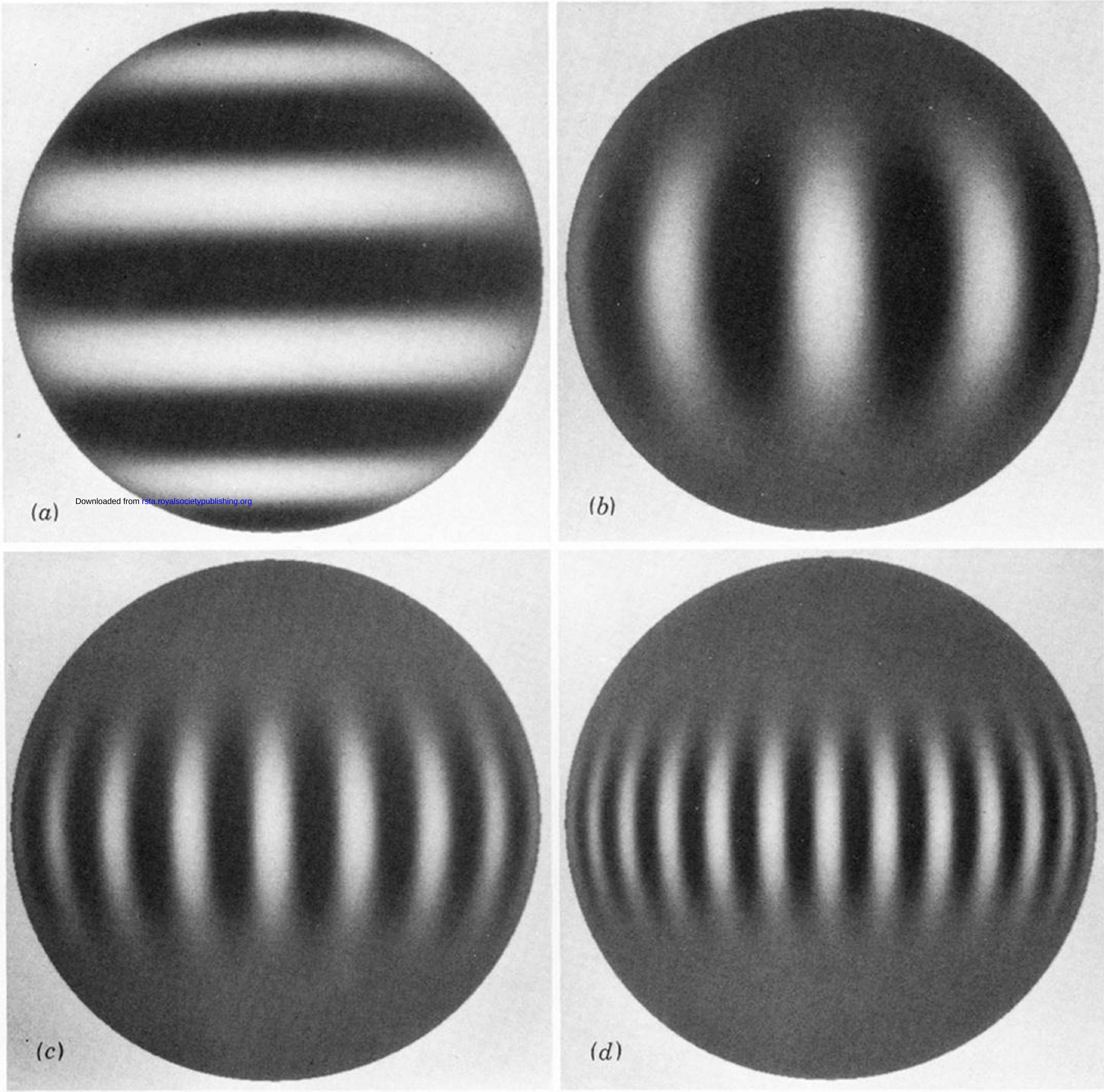


FIGURE 1. Contributions to the line-of-sight velocity at the surface of the Sun from a selection of five-minute oscillation modes. The grey level represents the magnitude of the velocity: black is towards and white is away from the observer (or vice-versa). The motion is almost vertical, so the mid-grey at the edges of the Sun's image represents zero velocity. (*a*) is the zonal mode ($m = 0$) of degree $l = 10$, and (*b*)–(*d*) are sectoral modes ($m = l$) of degrees 10, 20 and 30 respectively.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES